

File: SahooSharma-arXiv\_July2014.tex, printed: 14-7-2014, 6.14

# A NOTE ON A CLASS OF $p$ -VALENT STARLIKE FUNCTIONS OF ORDER BETA

SWADESH SAHOO\* AND NAVNEET LAL SHARMA

**ABSTRACT.** In this paper we obtain sharp coefficient bounds for certain  $p$ -valent starlike functions of order  $\beta$ ,  $0 \leq \beta < 1$ . Initially this problem was handled by Aouf in *M.K. Aouf, On a class of  $p$ -valent starlike functions of order  $\alpha$ , Internat. J. Math. & Math. Sci. 1987;10:733–744*. We pointed out that the proof given by Aouf was incorrect and a correct proof is presented in this paper.

**2010 Mathematics Subject Classification.** Primary 30C45; Secondary 30C55

**Key words.**  $p$ -valent analytic functions, starlike functions, differential subordination

## 1. INTRODUCTION

It is well-known that each univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  has the property  $|a_2| \leq 2$ , with equality occurring only for rotations of the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n.$$

This suggests the famous conjecture of Bieberbach [2], first proposed in 1916. This states that if  $f$  in the above form is univalent in  $\mathbb{D}$  then  $|a_n| \leq n$  for all  $n \geq 2$ . Initially this conjecture was proved in many special cases and has a long history. It was finally settled after several years by De Branges [4] in 1985. For basic theory of Bieberbach conjecture problem for number of classes of univalent functions we refer to [5, 9]. Part of this development, it was not generalized to the class of  $p$ -valent functions until 1948. The initiative was first taken by Goodman, see [8]. Similar problem for many other classes of  $p$ -valent functions can be found, for instance in [1, 7, 12]. In this paper we consider certain classes of  $p$ -valent functions in the unit disk and prove Bieberbach's conjecture for these functions.

For a natural number  $p$ , let  $\mathcal{A}_p$  denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

which are analytic and  $p$ -valent in the open unit disk.

Let  $g(z)$  and  $f(z)$  be analytic in  $\mathbb{D}$ . A function  $g(z)$  is called to be subordinate to  $f(z)$  if there exists an analytic function  $\phi(z)$  in  $\mathbb{D}$  with  $\phi(0) = 0$  and  $|\phi(z)| < 1$  ( $z \in \mathbb{D}$ ) such that  $g(z) = f(\phi(z))$ . We denote this subordination by  $g(z) \prec f(z)$  (see [11]).

---

\* The corresponding author.

Let  $\mathcal{S}_p(A, B, \beta)$  denote the class of functions  $f(z) \in \mathcal{A}_p$  satisfying

$$(1.2) \quad \frac{zf'(z)}{f(z)} \prec \frac{p + [pB + (A - B)(p - \beta)]z}{1 + Bz}, \quad z \in \mathbb{D}, \quad 0 \leq \beta < 1,$$

where  $A$  and  $B$  have the restriction  $-1 \leq B < A \leq 1$ . The class  $\mathcal{S}_p(A, B, \beta)$  was considered by Aouf in [1]. As a special case, we see that

$$\mathcal{S}_p(1, -1, \beta) = \mathcal{S}_p(\beta), \quad \mathcal{S}_1(\beta) = \mathcal{S}^*(\beta), \quad \mathcal{S}_p(0) = \mathcal{S}_p \text{ and } \mathcal{S}_1(A, B, 0) = \mathcal{S}^*(A, B).$$

Note that  $\mathcal{S}_p(\beta)$ , the class of  $p$ -valent starlike functions of order  $\beta$ , was studied by Goluzina in [7];  $\mathcal{S}^*(\beta)$ , the class of starlike functions of order  $\beta$  was introduced by Robertson in [13];  $\mathcal{S}_p$ , the usual class of  $p$ -valent starlike functions; and  $\mathcal{S}^*(A, B)$  was introduced by Janowski in [10].

Aouf estimated the coefficient bounds for the functions from the class  $\mathcal{S}_p(A, B, \beta)$  in [1] in which the proof is found to be incorrect. In this paper, we provide a correct proof.

## 2. Main result

The following Lemma is obtained by Goel and Mehrotra:

**Lemma 2.1.** [6, Theorem 1] *Let  $-1 \leq B < A \leq 1$  and  $f \in \mathcal{S}^*(A, B)$ . Then*

$$(2.1) \quad |a_2| \leq A - B;$$

for  $A - 2B \leq 1$ ,  $n \geq 3$ ,

$$(2.2) \quad |a_n| \leq \frac{A - B}{n - 1};$$

and for  $A - (n - 1)B > (n - 2)$ ,  $n \geq 3$ ,

$$(2.3) \quad |a_n| \leq \frac{1}{(n - 1)!} \prod_{j=2}^n (A - (j - 1)B).$$

The equality signs in (2.1) and (2.2) are attained for the functions

$$(2.4) \quad k_{n,A,B}(z) = \begin{cases} z(1 + B\delta z^{n-1})^{(A-B)/(n-1)B}, & \text{if } B \neq 0; \\ z \exp\left(\frac{A\delta z^{n-1}}{n-1}\right), & \text{if } B = 0, \end{cases} \quad |\delta| = 1,$$

and in (2.3) equality is attained for the functions

$$(2.5) \quad k_{A,B}(z) = \begin{cases} z(1 + B\delta z)^{(A-B)/B}, & \text{if } B \neq 0; \\ ze^{Az\delta}, & \text{if } B = 0, \end{cases} \quad |\delta| = 1.$$

However, a  $p$ -valent analog of Lemma 2.1 was wrongly proven by Aouf in the following form:

**Theorem A.** [1, Theorem 3] *Let  $-1 \leq B < A \leq 1$  and  $p \in \mathbb{N}$ . If  $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{S}_p(A, B, \beta)$ , then*

$$|a_n| \leq \prod_{j=0}^{n-p-1} \frac{|(B - A)(p - \beta) + Bj|}{j + 1}$$

for  $n \geq p + 1$ , and these bounds are sharp for all admissible  $A, B, \beta$  and for each  $n$ .

We now give the correct form of the statement stated in Theorem A and its proof.

**Theorem 2.2.** Let  $-1 \leq B < A \leq 1$  and  $p \in \mathbb{N}$ . If  $f(z) \in \mathcal{S}_p(A, B, \beta)$  is in the form (1.1), then we have

$$(2.6) \quad |a_{p+1}| \leq (A - B)(p - \beta);$$

for  $A(p - \beta) - B(p - \beta - 1) \leq 1$  (or  $A(p - \beta) - B(n - \beta - 1) \leq (n - p - 1)$ ),  $n \geq p + 2$ ,

$$(2.7) \quad |a_n| \leq \frac{(A - B)(p - \beta)}{n - p};$$

and for  $A(p - \beta) - B(n - \beta - 1) > (n - p - 1)$ ,  $n \geq p + 2$ ,

$$(2.8) \quad |a_n| \leq \prod_{j=1}^{n-p} \frac{(A(p - \beta) - B(p - \beta + j - 1))}{j}.$$

The inequalities (2.6), (2.7) and (2.8) are sharp.

*Proof.* Let  $f(z) \in \mathcal{S}_p(A, B, \beta)$ . By the relation (1.2) we can guarantee an analytic function  $\phi : \mathbb{D} \rightarrow \overline{\mathbb{D}}$  with  $\phi(0) = 0$  such that

$$\frac{zf'(z)}{f(z)} = \frac{p + [pB + (A - B)(p - \beta)]\phi(z)}{1 + B\phi(z)},$$

i.e.

$$zf'(z) - pf(z) = [(pB + (A - B)(p - \beta))f(z) - Bzf'(z)]\phi(z).$$

Substituting the series expansion (1.1), of  $f(z)$ , and canceling the factor  $z^p$  on both sides, we obtain

$$\sum_{k=1}^{\infty} ka_{p+k}z^k = \left( (A - B)(p - \beta) - \sum_{k=1}^{\infty} [B(p + k) + (-pB + (B - A)(p - \beta))]a_{p+k}z^k \right) \phi(z).$$

Rewriting it, we get

$$\sum_{k=1}^{\infty} ka_{p+k}z^k = \left( (A - B)(p - \beta) + \sum_{k=1}^{\infty} [A(p - \beta) - B(k + p - \beta)]a_{p+k}z^k \right) \phi(z).$$

By Clunie's method [3] (for instance see [15, 14]) for  $n \in \mathbb{N}$ , we observe that

$$\sum_{k=1}^n k^2 |a_{p+k}|^2 \leq (A - B)^2(p - \beta)^2 + \sum_{k=1}^{n-1} [A(p - \beta) - B(k + p - \beta)]^2 |a_{p+k}|^2.$$

Simplification of the above inequality leads to

$$|a_{p+n}|^2 \leq \frac{1}{n^2} \left( (A - B)^2(p - \beta)^2 + \sum_{k=1}^{n-1} \left( [A(p - \beta) - B(k + p - \beta)]^2 - k^2 \right) |a_{p+k}|^2 \right)$$

or

$$|a_{p+n}|^2 \leq \frac{1}{n^2} \left( (A - B)^2(p - \beta)^2 + \sum_{k=2}^n \left( [A(p - \beta) - B(k + p - \beta - 1)]^2 - (k - 1)^2 \right) |a_{p+k-1}|^2 \right).$$

Above inequality can be rewritten by replacing  $p + n$  by  $n$  as

$$(2.9) \quad |a_n|^2 \leq \frac{1}{(n-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p} \left( [A(p-\beta) - B(k+p-\beta-1)]^2 - (k-1)^2 \right) |a_{p+k-1}|^2 \right)$$

for  $n \geq p+1$ .

Note that the terms under the summation in the right hand side of (2.9) may be positive as well as negative. We investigate it by including here a table (see Table 1) for values of  $W := (A(p-\beta) - B(k+p-\beta-1))^2 - (k-1)^2$  for various choices of  $A, B, k, \beta$  and  $p$ .

<b>k</b>	<b>p</b>	<b>A</b>	<b>B</b>	<b><math>\beta</math></b>	<b>W</b>
2	1	0.8	0.5	0	-0.96
2	1	-0.5	-0.8	0	0.21
3	2	0.5	0.4	0.5	-3.5775
3	2	-0.1	-0.7	0.5	1.29

Table 1

*(This the place where the incorrectness of Aouf's proof is found!)*

So, we can not apply direct mathematical induction in (2.9) to establish the required bounds for  $|a_n|$ . Therefore, we are considering different cases for this.

First, for  $n = p+1$ , we easily see that (2.9) reduces to

$$|a_{p+1}| \leq (A-B)(p-\beta)$$

which establishes (2.6).

Secondly,  $A(p-\beta) - B(p-\beta-1) \leq 1$  if and only if  $A(p-\beta) - B(n-\beta-1) \leq (n-p-1)$  for  $n \geq p+2$ . Since all the terms under the summation in (2.9) are non-positive, we reduce to

$$|a_n| \leq \frac{(A-B)(p-\beta)}{n-p}$$

for  $A(p-\beta) - B(p-\beta+1) \leq 1$ ,  $n \geq p+2$ . This proves (2.7). The equality holds in (2.6) and (2.7) for the functions

$$k_{n,A,B,p}(z) = \begin{cases} z^p (1 + B\delta z^{n-1})^{(A-B)(p-\beta)/(n-1)B}, & B \neq 0; \\ z^p \exp\left(\frac{\Lambda(p-\beta)\delta z^{n-1}}{n-1}\right), & B = 0, \end{cases} \quad |\delta| = 1.$$

Finally let us prove (2.8) when  $A(p-\beta) - B(n-\beta-1) > (n-p-1)$ ,  $n \geq p+2$ . We see that all the terms under the summation in (2.9) are positive. We prove the inequality by the usual mathematical induction. Fix  $n$ ,  $n \geq p+2$  and suppose that (2.8) holds for

$k = 3, 4, \dots, n - p$ . Then from (2.9), we find

$$(2.10) \quad |a_n|^2 \leq \frac{1}{(n-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p} \left( [A(p-\beta) - B(k+p-\beta-1)]^2 - (k-1)^2 \right) \prod_{j=1}^{k-1} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} \right).$$

It is now enough to show that the square of the right hand side of (2.8) is equal to the right hand side of (2.10), that is

$$(2.11) \quad \prod_{j=1}^{m-p} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} = \frac{1}{(m-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{m-p} \left( [A(p-\beta) - B(k+p-\beta-1)]^2 - (k-1)^2 \right) \prod_{j=1}^{k-1} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} \right)$$

for  $A(p-\beta) - B(m-\beta-1) > (m-p-1)$ ,  $m \geq p+2$ . We also use the induction principle to prove (2.11).

The equation (2.11) is recognized for  $m = p+2$ . Suppose that (2.11) is true for all  $m$ ,  $p+2 < m \leq n-p$ . Then from (2.10), we obtain

$$\begin{aligned} |a_n|^2 \leq \frac{1}{(n-p)^2} & \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p-1} \left( [A(p-\beta) - B(k+p-\beta-1)]^2 - (k-1)^2 \right) \right. \\ & \times \prod_{j=1}^{k-1} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} + \left( [A(p-\beta) - B(n-\beta-1)]^2 \right. \\ & \left. \left. - (n-p-1)^2 \right) \times \prod_{j=1}^{n-p-1} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} \right). \end{aligned}$$

Using the induction hypothesis, for  $m = n-1$ , we get

$$\begin{aligned} |a_n|^2 \leq \frac{1}{(n-p)^2} & \left( (n-p-1)^2 \prod_{j=1}^{n-p-1} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} \right. \\ & \left. + \left( [A(p-\beta) - B(n-\beta-1)]^2 - (n-p-1)^2 \right) \prod_{j=1}^{n-p-1} \frac{[A(p-\beta) - B(p-\beta+j-1)]^2}{j^2} \right). \end{aligned}$$

Hence

$$|a_n| \leq \prod_{j=1}^{n-p} \frac{[A(p-\beta) - B(p-\beta+j-1)]}{j}.$$

It is easy to prove that the bounds are sharp for the function

$$k_{A,B,p}(z) = \begin{cases} z^p(1 + B\delta z)^{(A-B)(p-\beta)/B}, & B \neq 0; \\ z^p e^{A(p-\beta)z\delta}, & B = 0, \end{cases} \quad |\delta| = 1.$$

This completes the proof of Theorem 2.2.  $\square$

We remark that, choosing  $p = 1$  and  $\beta = 0$  in Theorem 2.2 we turned into Lemma 2.1.

**Acknowledgements.** The second author acknowledges the support of National Board for Higher Mathematics, Department of Atomic Energy, India (grant no. 2/39(20)/2010-R&D-II).

## REFERENCES

- [1] M. K. AOUF, On a class of  $p$ -valent starlike functions of order  $\alpha$ , *Internat. J. Math. & Math. Sci.*, **10** (1987), 733–744.
- [2] L. BIEBERBACH, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *S.-B. Preuss. Akad. Wiss.*, 1916, 940–955.
- [3] J. CLUNIE, On meromorphic schlicht functions, *J. London Math. Soc.*, **34** (1959), 215–216.
- [4] L. DE BRANGES, A proof of the Bieberbach conjecture, *Acta Math.*, **154** (1985), 137–152.
- [5] P. L. DUREN, *Univalent Functions*, Springer-Verlag, 1983.
- [6] R. M. GOEL AND B. S. MEHROK, On the coefficients of a subclass of starlike functions, *Indian J. Pure Appl. Math.*, **12**(5) (1981), 634–647.
- [7] E. G. GOLUZINA, On the coefficients of a class of functions, regular in a disk and having an integral representation in it, *J. of Soviet Math.*, **6** (1974), 606–617.
- [8] A. W. GOODMAN, On some determinants related to  $p$ -valent functions, *Trans. Amer. Math. Soc.*, **63** (1948), 175–192.
- [9] A. W. GOODMAN, *Univalent Functions*, Vol. 12, Mariner, Tampa, Florida, 1983.
- [10] W. JANOWSKI, Some extremal problem for certain families of analytic functions, *Ann. Polon. Math.*, **28** (1973), 297–326.
- [11] S. S. MILLER AND P. T. MOCANU, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, New York and Basel, Marcel Dekker, 2000.
- [12] D. A. PATIL AND N. K. THAKARE, On convex hulls and extreme points of  $p$ -valent starlike and convex classes with applications, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)* **27**(75) (1983), 145–160.
- [13] M. S. ROBERTSON, On the theory of univalent functions, *Annals of Mathematics*, **37** (1936), 374–408.
- [14] M. S. ROBERTSON, Quasi-subordination and coefficient conjectures, *J. Bull. Amer. Math. Soc.*, **76** (1970), 1–9.
- [15] W. ROGOSINSKI, On the coefficients of subordinate functions, *Proc. London Math. Soc.*, **48**(2) (1943), 48–82.

SWADESH SAHOO, DISCIPLINE OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY INDORE, INDORE 452 017, INDIA

*E-mail address:* swadesh@iiti.ac.in

NAVNEET LAL SHARMA, DISCIPLINE OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY INDORE, INDORE 452 017, INDIA

*E-mail address:* sharma.navneet23@gmail.com